Computer Graphics
Lecture-06
Two –Dimensional Transformations

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Lecturer

DIIT

11/15/2021

Matrix Math

- Why do we use matrix?
  - More convenient organization of data.
  - More efficient processing
  - Enable the combination of various concatenations
- Matrix addition and subtraction

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \pm \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \pm \mathbf{c} \\ \mathbf{b} \pm \mathbf{d} \end{pmatrix}$$

Matrix Math

- Matrix Multiplication
  - Dot product

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} \end{pmatrix} = \begin{pmatrix} \mathbf{a.e + b.g} & \mathbf{a.f + b.h} \\ \mathbf{c.e + d.g} & \mathbf{c.f + d.h} \end{pmatrix}$$

Matrix Math

- What about this?

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \bullet \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \textbf{No!!}$$

- Type of matrix

$$\begin{pmatrix} a & b \end{pmatrix} \qquad \begin{pmatrix} a \\ b \end{pmatrix}$$

Row-vector **Column-vector**

Matrix Math

- Is there a difference between possible representations?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \end{bmatrix}$$

$$\begin{bmatrix} e & f \end{bmatrix} \bullet \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ae+cf & be+df \end{bmatrix}$$

$$\begin{bmatrix} e & f \end{bmatrix} \bullet \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ae+bf & ce+df \end{bmatrix}$$

Matrix Math

- The column-vector representation for a point.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cx + Dy \end{bmatrix}$$

Rotation

- Example
  - Find the transformed point, P', caused by rotating P= (5, 1) about the origin through an angle of 90°.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cdot\cos\theta - y\cdot\sin\theta \\ x\cdot\sin\theta + y\cdot\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot\cos 90 - 1\cdot\sin 90 \\ 5\cdot\sin 90 + 1\cdot\cos 90 \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot 0 - 1\cdot 1 \\ 5\cdot 1 + 1\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Composite Transformation

- We can represent any sequence of transformations as a single matrix.
  - Composite transformations – matrix • matrix.
- Composite transformations:
  - Rotate about an arbitrary point – translate, rotate, translate
  - Scale about an arbitrary point – translate, scale, translate
  - Change coordinate systems – translate, rotate, scale
- Does the order of operations matter?

Composition Properties

- Is matrix multiplication associative?
  - (A.B).C $\stackrel{?}{=}$ A.(B.C)

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \bullet \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \bullet \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$$= \begin{bmatrix} aei+bgi+afk+bhk & aej+bgj+afl+bhl \\ cei+dgi+cfk+dhk & cej+dgj+cfl+dhl \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \bullet \begin{bmatrix} i & j \\ k & l \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{bmatrix}$$

$$= \begin{bmatrix} aei+afk+bgi+bhk & aej+afl+bgj+bhl \\ cei+cfk+dgi+dhk & cej+cfl+dgj+dhl \end{bmatrix}$$

Composition Properties

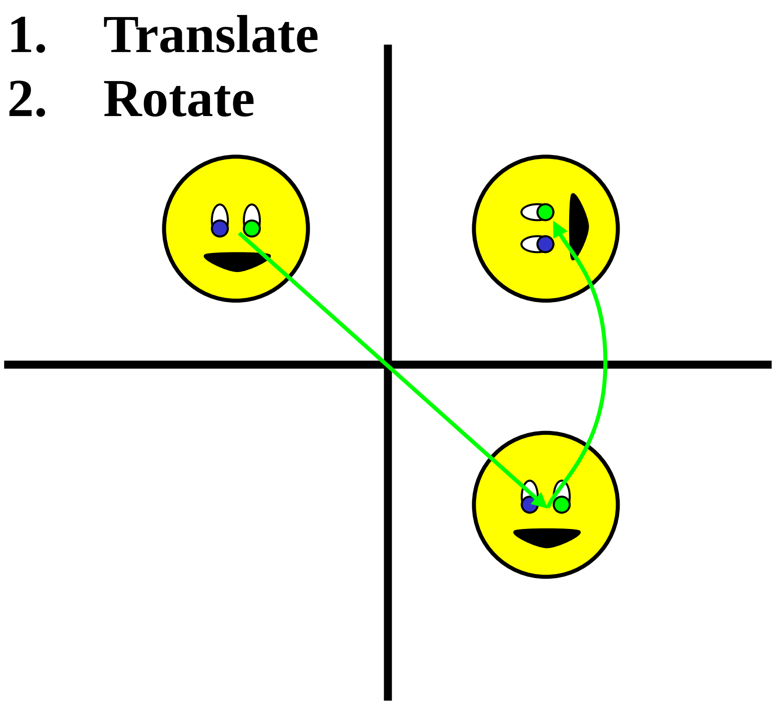
- Is matrix multiplication commutative?
  - A . B $\overset{?}{=}$ B . A

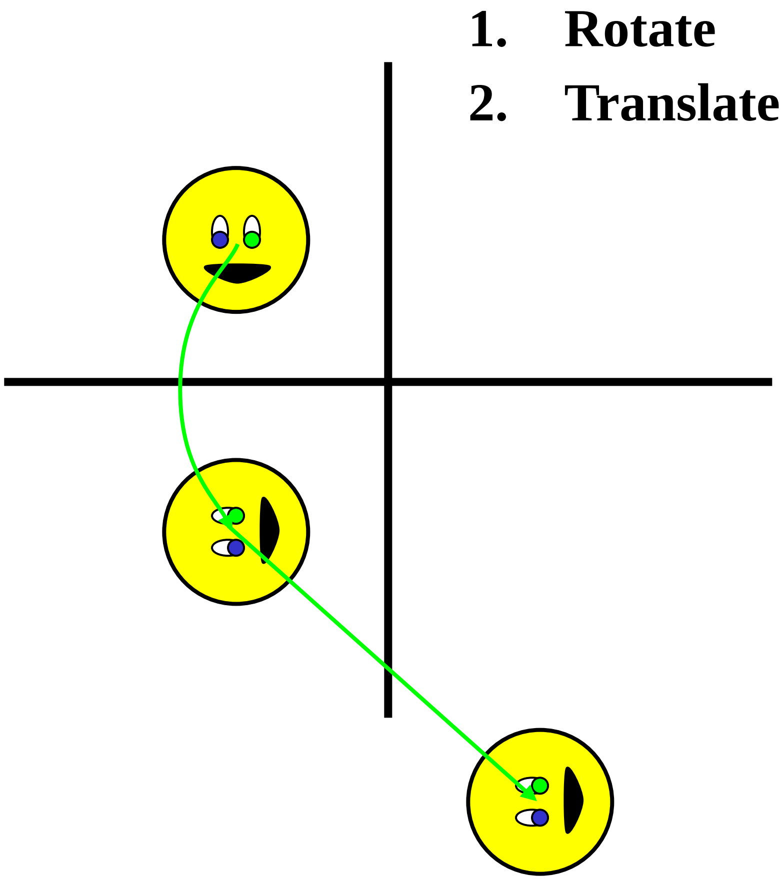
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \bullet \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{bmatrix}$$

Order of operations

So, it does matter. Let's look at an example:





Composite Transformation Matrix

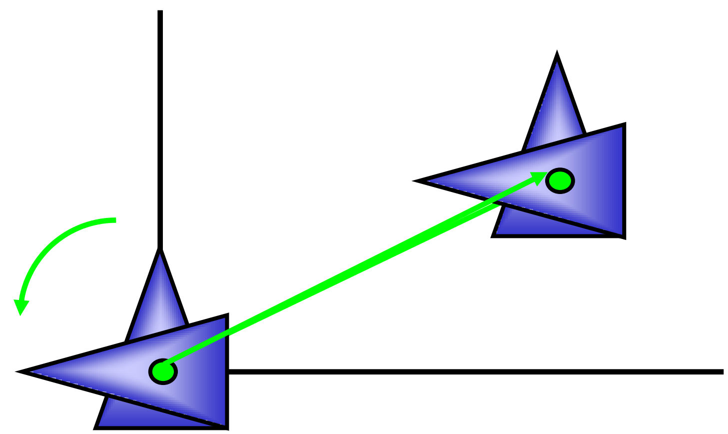
- **Arrange the transformation matrices in order from right to left.**
- **General Pivot- Point Rotation**
  - **Operation :-**
    1. **Translate (pivot point is moved to origin)**
    2. **Rotate about origin**
    3. **Translate (pivot point is returned to original position)**

T(pivot) • R(θ) • T(–pivot)

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} \cos\theta & -\sin\theta & -t_x\cos\theta + t_y\sin\theta \\ \sin\theta & \cos\theta & -t_x\sin\theta - t_y\cos\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & -t_x\cos\theta + t_y\sin\theta + t_x \\ \sin\theta & \cos\theta & -t_x\sin\theta - t_y\cos\theta + t_y \\ 0 & 0 & 1 \end{pmatrix}$$



Composite Transformation Matrix

- Example
  - Perform 60° rotation of a point P(2, 5) about a pivot point (1,2). Find P'?

$$\begin{pmatrix} \cos\theta & -\sin\theta & -t_x\cos\theta + t_y\sin\theta + t_x \\ \sin\theta & \cos\theta & -t_x\sin\theta - t_y\cos\theta + t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{1} \end{pmatrix}$$

Sin 60 = 0.8660

Cos 60 = 1/2

$$\begin{pmatrix} 0.5 & -0.866 & -1.0.5 + 2.0.866 + 1 \\ 0.866 & 0.5 & -1.0.866 - 2.0.5 + 2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} 0.5 & -0.866 & 2.232 \\ 0.866 & 0.5 & 0.134 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{-1.098} \\ \mathbf{4.366} \\ \mathbf{1} \end{pmatrix}$$

P' = (-1, 4)

Composite Transformation Matrix

General Fixed-Point Scaling

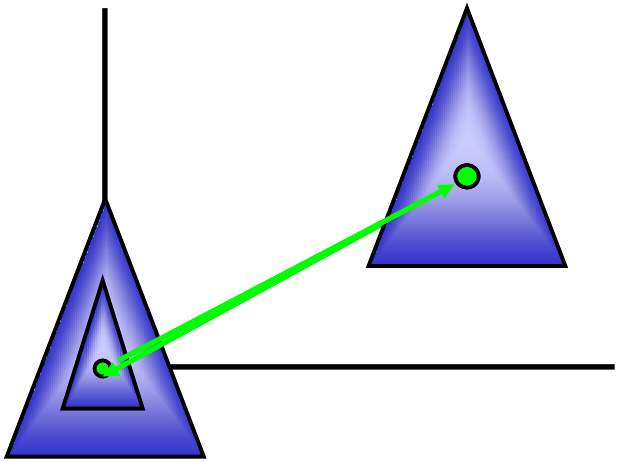
Operation :-

1. Translate (fixed point is moved to origin)
2. Scale with respect to origin
3. Translate (fixed point is returned to original position)

T(fixed) • S(scale) • T(–fixed)

Find the matrix that represents scaling of an object with respect to any fixed point?

Given P(6, 8) , Sx = 2, Sy = 3 and fixed point (2, 2). Use that matrix to find P'?



Answer

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} Sx & 0 & -t_x Sx \\ 0 & Sy & -t_y Sy \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} Sx & 0 & -t_x Sx + t_x \\ 0 & Sy & -t_y Sy + t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$x = 6$, $y = 8$, $Sx = 2$, $Sy = 3$, $t_x = 2$, $t_y = 2$

$$\begin{pmatrix} 2 & 0 & -2(2) + 2 \\ 0 & 3 & -2(3) + 2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{6} \\ \mathbf{8} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{10} \\ \mathbf{20} \\ \mathbf{1} \end{pmatrix}$$

Composite Transformation Matrix

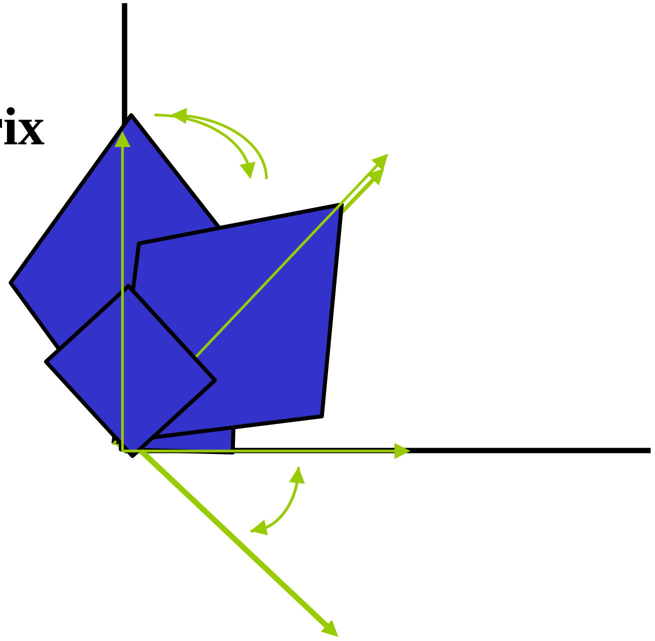
General Scaling Direction

Operation :-

1. **Rotate (scaling direction align with the coordinate axes)**
2. **Scale with respect to origin**
3. **Rotate (scaling direction is returned to original position)**

R(–θ) • S(scale) • R(θ)

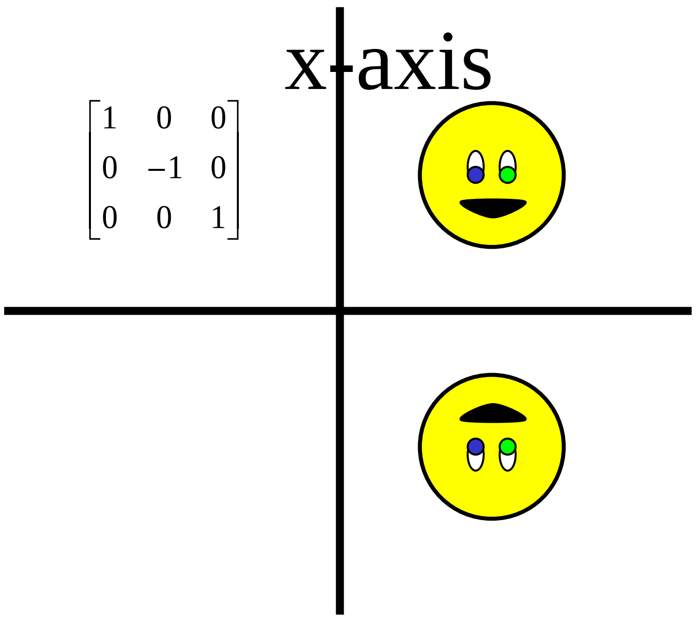
Find the composite transformation matrix by yourself !!



Other transformations

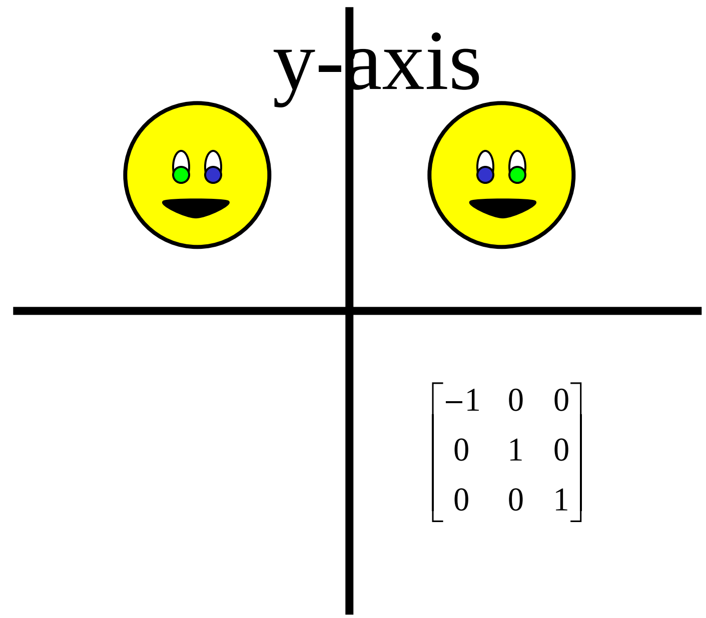
Reflection:

x-axis



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

y-axis



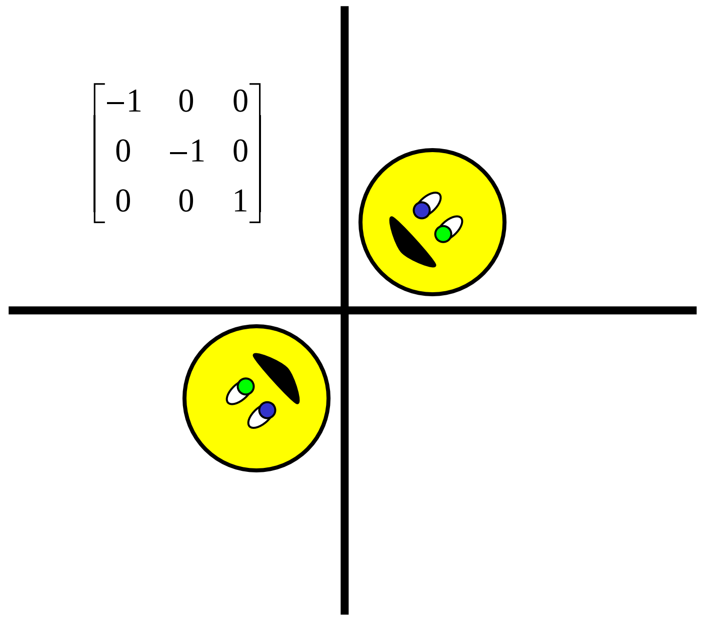
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Other transformations

Reflection:

origin

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

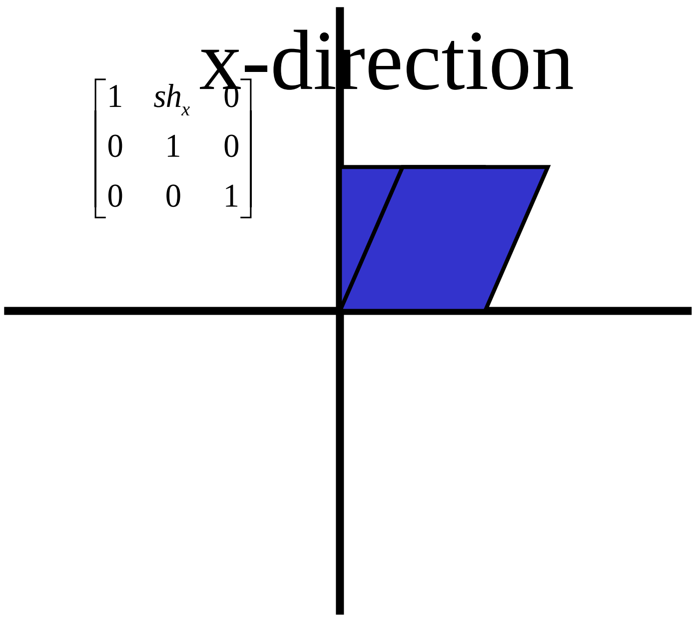


Other transformations

Shear:

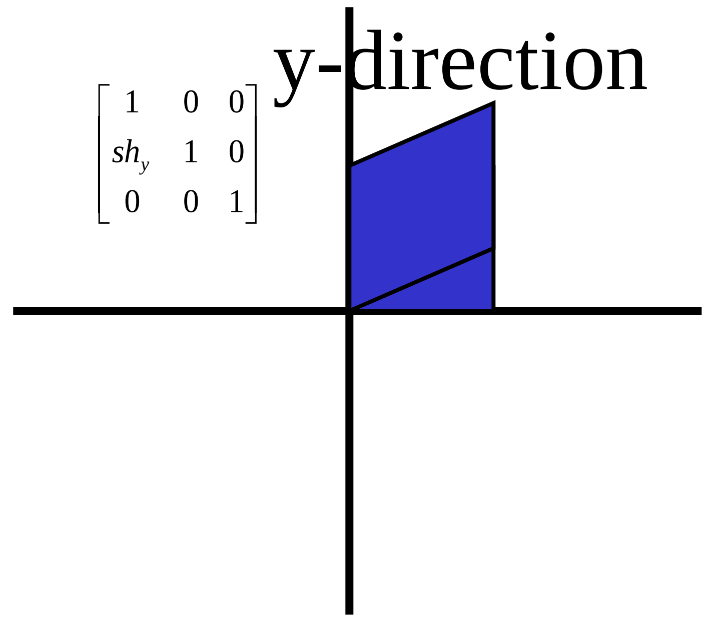
x-direction

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



y-direction

$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Another Example.